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Stratified disordered media: exact solutions for transport parameters and their self-averaging properties

M Clincy[1] **and H Kinzelbach**[2]

[1] Department of Physics and Astronomy, University of Edinburgh, Edinburgh EH9 3JZ, UK
[2] Institut für Theoretische Physik, Philosophenweg 19, 69120 Heidelberg, Germany

E-mail: m.clincy@ed.ac.uk and kibach@tphys.uni-heidelberg.de

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Abstract

We investigate the transport of a passive tracer in a two-dimensional stratified random medium with flow parallel and perpendicular to the strata. Assuming a Gaussian random flow with a Gaussian correlation function, it is not only possible to derive exact expressions for the temporal behaviour of the dispersion coefficients characterizing the tracer transport, but also to investigate the self-averaging properties of these quantities. We give explicit results for the dispersion coefficient and its mean square fluctuations as a function of time. As it turns out, the sample-to-sample fluctuations which are encoded in the latter quantity remain finite even in the asymptotic limit of infinite times, which implies that the given two-dimensional model is not self-averaging.

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1. Introduction

Twenty years after the investigation of transport in stratified random media by Matheron and de Marsily [1], their model is still intriguing, not only because it is exactly solvable, but also because it serves as a prototype of models exhibiting super-diffusive behaviour [2]. As such it has found applications in various fields like hydromechanics [3, 4] and polymer physics [5, 6].

The original model can be described as follows: consider a two-dimensional random medium consisting of distinct parallel layers (or strata) which extend in x_1-direction and have varying permeability. The resulting velocity field $\mathbf{u}$ for a fluid flow in this medium depends only on the component x_2 transverse to the layers. In a random ensemble of such stratified media, it is a random field with given statistical properties. The properties of such two-dimensional stratified models are well known in the asymptotic limit of large times, $t \to \infty$, and for the case of vanishing local diffusion, see e.g. [7–9]. The large-scale transport behaviour is roughly characterized by the mean square displacement of the tracer particles, appropriately averaged over the ensemble of the random flow fields. The time derivative of this quantity encodes the large-scale diffusion or, as it is called, 'dispersion' coefficient $D(t)$ of the tracer cloud as a function of time. The transport is called diffusive if this coefficient becomes a finite constant in the asymptotic long-time limit $t \to \infty$ and is called super-diffusive if it asymptotically grows with time as $D(t) \sim t^{\alpha}$ with $\alpha > 0$. For a flow field parallel to the strata the model exhibits (for $t \to \infty$) such a super-diffusive behaviour with $\alpha = 1/2$. In the more general case, where one also has a transversal flow component, the dispersion coefficient eventually tends to a constant value, i.e. the transport shows normal diffusive behaviour.

However, what is still lacking, to the best of our knowledge, is a detailed analysis of the pre-asymptotic behaviour of the model and, even more interestingly, of the corresponding sample-to-sample fluctuations of the dispersion coefficient. The latter is crucial for the reliability of the model. Small fluctuations indicate that the transport parameters which are constructed as averages over the random ensemble are indeed also representative of the behaviour of individual realizations of the ensemble.

In the present paper, we try to close this gap. Assuming a stratified Gaussian random flow with a Gaussian correlation function, we derive exact expressions for the dispersion coefficient and its mean square fluctuations as a function of time. The latter quantity encodes the sample-to-sample fluctuations of the dispersion coefficient. We show that it remains finite even in the asymptotic limit of infinite times $t \to \infty$ as long as the initial tracer distribution is of a finite extent which implies that in the given two-dimensional model the dispersion coefficient, surprisingly, is not self-averaging.

The paper is organised as follows. In section 2 we give a brief description of the two-dimensional Matheron–de Marsily model and define the corresponding transport parameters. The next two sections deal with the exactly stratified model (i.e. a model where the flow field is aligned parallel to the strata) and with the more general case of a non-vanishing transverse flow component. In section 5 we analyse the self-averaging properties of the given transport parameters.

2. The model

The transport of a passive tracer particle by a two-dimensional flow field $\mathbf{u}(\mathbf{x})$ is modelled by a Langevin equation $\mathrm{d}\mathbf{x}(t)/\mathrm{d}t = \mathbf{u}(\mathbf{x}) + \boldsymbol{\xi}(t)$, where $\mathbf{x}(t)$ denotes the position vector of the particle and $\boldsymbol{\xi}(t)$ is a Gaussian white noise which generates the local diffusion process. It has zero mean $\langle \xi_i(t) \rangle = 0$ (with $i = 1, 2$) and a correlation function $\langle \xi_i(t)\xi_j(t') \rangle = 2D_{ij}\,\delta(t-t')$, where $\langle \cdots \rangle$ stands for the average over the white noise ensemble. We assume a diagonal local diffusion tensor, $D_{ij} = \delta_{ij}\, D_i$, which, however, may have different entries for the longitudinal and the transversal diffusion constants, $D_1 \neq D_2$.

In a two-dimensional stratified medium the flow field $\mathbf{u}$ depends on x_2 only. In the model investigated here, one has $\mathbf{u}(\mathbf{r}) = (u_0 + u'(x_2),\ v_0)$, where u_0 and v_0 denote the components of a constant drift velocity in the 1- and 2-directions, whereas $u'(x_2)$ is the random flow contribution generated by the layer-to-layer permeability variations (see e.g. [4]).

The Langevin equation therefore reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}x_1(t) = u_0 + u'(x_2(t)) + \xi_1(t) \tag{2.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}x_2(t) = v_0 + \xi_2(t). \tag{2.2}$$

The flow contribution $u'(x_2(t))$ is a random function chosen from an appropriate stationary ensemble which defines its stochastic properties. In the model investigated here, we assume that it has zero mean and a Gaussian correlation function,

$$\overline{u'(x_2)} = 0 \tag{2.3}$$

$$\overline{u'(x_2)u'(x_2')} = \sigma_0^2 \exp\left(-\frac{(x_2 - x_2')^2}{2l^2}\right). \tag{2.4}$$

The over-bar denotes the average with respect to disorder ensemble, the parameter σ_0 quantifies the disorder strength, and l denotes the correlation length of the field. As is shown below, these two relations uniquely determine the temporal behaviour of the dispersion coefficient; higher correlation functions do not enter. This, however, is no longer true if one deals with the sample-to-sample fluctuations of this dispersion coefficient. In this case one also needs an explicit expression for the four-point function of the random field. For simplicity, and without severe restriction of generality, let us assume a Gaussian ensemble, so that all higher correlation functions can be decomposed into products of the given two-point function.

Equations (2.1) and (2.2) define a system of differential equations which is readily solved by

$$x_1(t)|_{\mathbf{x}_0} = x_{01} + u_0 t + \int_0^t \mathrm{d}t' u'(x_2(t')) + \int_0^t \mathrm{d}t' \xi_1(t') \tag{2.5}$$

$$x_2(t)|_{\mathbf{x}_0} = x_{02} + v_0 t + \int_0^t \mathrm{d}t' \xi_2(t'). \tag{2.6}$$

The single tracer particle subject to a given noise history is characterized by its trajectory $\mathbf{x}(t)|_{\mathbf{x}_0}$, where the subscript $\mathbf{x}_0$ indicates the fact that the tracer coordinates at a given time t still depend on the initial injection position $\mathbf{x}(t = 0)|_{\mathbf{x}_0} \equiv \mathbf{x}_0 = (x_{01}, x_{02})$. In the general case where one has a spatially extended injection of material at time $t = 0$ the injection points $\mathbf{x}_0$ are distributed according to a distribution $\rho(\mathbf{x}_0)$ which without restriction is chosen to be normalized,

$$\int \mathrm{d}^2 x_0\, \rho(\mathbf{x}_0) = 1. \tag{2.7}$$

The shape of the resulting concentration distribution is characterized by its moments

$$\langle x_i(t)^n \rangle \equiv \int \mathrm{d}^2 x_0\, \rho(\mathbf{x}_0) \langle x_i(t)|_{\mathbf{x}_0}^n \rangle \tag{2.8}$$

(with $n = 1, 2, \ldots$) which are used, in particular, to define the centre-of-mass velocity and the dispersion coefficients of the resulting concentration plume. The centre-of-mass velocity (for a given realization of the flow field) reads

$$u_i(t) = \frac{\mathrm{d}}{\mathrm{d}t} \langle x_i(t) \rangle. \tag{2.9}$$

The dispersion coefficients which give the spreading rate of the plume are defined by

$$D_{ij}(t) = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \{\langle x_i(t)\, x_j(t) \rangle - \langle x_i(t) \rangle \langle x_j(t) \rangle\}. \tag{2.10}$$

In a stochastic approach, the given single medium is viewed as one particular realization of a spatial stochastic process. Large-scale transport coefficients are derived from appropriately constructed averages over the ensemble of all possible medium realizations. These averages represent statistical properties of the (artificial) ensemble as a whole. At first glance, they therefore seem to be of limited predictive value with respect to the properties of a single given aquifer. For appropriately chosen quantities, however, in many cases the fluctuations from realization to realization become small as soon as the cloud has sampled a sufficiently large representative part of the given medium. As soon as the transport properties found in different realizations of the medium then fluctuate only weakly around the corresponding ensemble averages, these averages also predict properties characteristic of a single typical realization of the medium.

In the following discussion, we investigate such ensemble averages of the centre-of-mass velocity and the dispersion coefficients. We call the averaged quantities corresponding to (2.9) and (2.10) 'effective' quantities. The resulting effective centre-of-mass velocity and effective dispersion coefficient are defined as

$$u_i^{\mathrm{eff}}(t) \equiv \overline{u_i(t)} = \frac{\mathrm{d}}{\mathrm{d}t}\overline{\langle x_i(t)\rangle} \tag{2.11}$$

and

$$D_{ii}^{\mathrm{eff}}(t) \equiv \overline{D_{ii}(t)} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\overline{\langle x_i^2(t)\rangle} - \overline{\langle x_i(t)\rangle^2}\right\}. \tag{2.12}$$

The non-diagonal dispersion coefficients vanish due to reasons of symmetry. Note that two types of averages are involved: the average over the white noise which generates the local diffusion process, indicated by the brackets, and the average over the disorder ensemble, indicated by the over-bar. The order in which these averages are performed is crucial, as the following slightly different definition of an averaged dispersion coefficient demonstrates. We define what we call 'ensemble dispersion coefficients' as

$$D_{ii}^{\mathrm{ens}}(t) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\overline{\langle x_i^2(t)\rangle} - \overline{\langle x_i(t)\rangle}^2\right\}. \tag{2.13}$$

This quantity, which is also well known and frequently used in the literature, represents the dispersion characteristics of the whole ensemble of realizations of the flow field. It takes into account an artificial dispersion effect caused by fluctuations of the centre-of-mass positions of the tracer distributions in different realizations of the inhomogeneous medium. This effect is suppressed in the effective dispersion coefficient as defined in equation (2.12) because there the centre-of-mass positions are superimposed before the ensemble average is performed. For an extensive discussion of these problems and the related literature, see [10, 11]. In general, the experimentally observable dispersion which is a property related to a single given realization is represented by the effective quantity $D^{\mathrm{eff}}(t)$. Usually, at finite times the ensemble quantity overestimates the experimentally observable dispersion considerably (see again the detailed discussion given in [10, 11]). In the asymptotic long-time limit $t \to \infty$, however, one usually expects that the two types of dispersion coefficients become equal.

For the model investigated here, the full temporal behaviour of the given transport coefficients can be evaluated explicitly. The corresponding calculations are elementary but somewhat tedious. They are summarized in appendix A. Using the explicit solution (2.6) and performing the appropriate averages, one finds the more or less trivial results

$$u_1^{\mathrm{eff}}(t) = u_0 \qquad u_2^{\mathrm{eff}}(t) = v_0 \qquad \text{and} \qquad D_{22}^{\mathrm{ens}}(t) = D_{22}^{\mathrm{eff}}(t) = D_2 \tag{2.14}$$

valid for all times. A more interesting behaviour follows for the longitudinal dispersion coefficients. Here one derives the relations

$$D_{11}^{\mathrm{ens}}(t) = D_1 + \frac{1}{2}\partial_t \int_k \int_{k'} \overline{\tilde{u}(k)\tilde{u}(k')} \times \int_0^t \mathrm{d}t' \int_0^t \mathrm{d}t'' \mathrm{e}^{-(k^2 D_2 t' + k'^2 D_2 t'' + 2kk' D_2 \min(t',t''))} \tag{2.15}$$

$$D_{11}^{\mathrm{eff}}(t) = D_{11}^{\mathrm{ens}}(t) - \frac{1}{2}\partial_t \overline{\left(\int_k \tilde{u}(k) \int \mathrm{d}^2 x_0 \rho(\mathbf{x}_0) \mathrm{e}^{-\mathrm{i}k y_0} \int_0^t \mathrm{d}t' \mathrm{e}^{-k^2 D_2 t'}\right)^2} \tag{2.16}$$

where we use $\int_k \cdots \equiv (2\pi)^{-2} \int \mathrm{d}^2 k \cdots$ as a convenient shorthand notation.

One can already observe three simple features. First of all, the effective and the ensemble dispersion coefficients in the direction transversal to the flow field are both equal to the local dispersion coefficient D_2. The heterogeneities of the model do not change this property because they do not depend on the fluctuating part of the flow field. The particles perform a simple random walk in the x_2-direction. Secondly, the effective dispersion coefficient in the x_1-direction does not depend on the longitudinal extension of the initial distribution. This stems from the infinite correlation length of the flow field in that direction. The third observation is that the ensemble dispersion coefficient does not show any dependence on the size of the initial distribution.

What remains to be evaluated is the explicit behaviour of the longitudinal dispersion coefficient. In the following we will discuss the cases $v_0 = 0$, where the fluid flow is aligned with the strata of the medium (the so-called 'exactly stratified model'), and $v_0 \neq 0$ separately as they give rise to a different asymptotic behaviour. We will further distinguish between a point-like initial distribution at $x_{10} = x_{20} = 0$ (which implies $\rho(\mathbf{x}_0) = \delta(\mathbf{x}_0)$ in (2.8)) and a spatially extended distribution

$$\rho(\mathbf{x}_0) = \frac{1}{\sqrt{2\pi}L} \exp\left(-\frac{x_{02}^2}{2L^2}\right)\delta(x_{01}) \tag{2.17}$$

where the injection points are aligned along a line in the x_2-direction. As it turns out, the line shape is no restriction of generality since a finite extent in the x_1-direction does not alter the given results.

3. The exactly stratified medium

For $v_0 = 0$ and a point-like initial distribution the expressions for the effective dispersion coefficients given by equations (2.15) and (2.16) can be evaluated easily. One derives

$$D_{11}^{\mathrm{ens}}(t) = D_1 + \sigma_0^2 \tau_D \left(\sqrt{1 + \frac{2t}{\tau_D}} - 1\right) \tag{3.1}$$

$$D_{11}^{\mathrm{eff}}(t) = D^{\mathrm{ens}}(t) - \sigma_0^2 \tau_D \left(\sqrt{1 + \frac{4t}{\tau_D}} - \sqrt{1 + \frac{2t}{\tau_D}}\right) \tag{3.2}$$

where the timescale $\tau_D = l^2/D_2$ is a measure of the diffusive transport over one transversal correlation length. The result is plotted in figure 1. There are two characteristic regimes. For $t \ll \tau_D$ the dispersion coefficients asymptotically are given as

$$D_{11}^{\mathrm{ens}}(t) = D_1 + \sigma_0^2 t + O\left((t/\tau_D)^2\right) \tag{3.3}$$

$$D_{11}^{\mathrm{eff}}(t) = D_1 + O\left((t/\tau_D)^2\right). \tag{3.4}$$

In this regime there exist great fluctuations between the centre-of-mass coordinates of the source in different realizations of the flow field. The transport is dominated by advection which explains the linear time dependence of the ensemble dispersion coefficient. In each realization the particles only experience local diffusion which can be seen in the effective value which is approximately constant. This behaviour characterizes the Taylor regime as already described in [3].

In the opposite long-time limit, $t \gg \tau_D$, the dispersion coefficients take the following asymptotic form:

$$D_{11}^{\mathrm{ens}}(t) = D_1 + \sigma_0^2 \sqrt{2\tau_D}\, t^{1/2} + O\left((t/\tau_D)^{-1/2}\right) \tag{3.5}$$

$$D_{11}^{\mathrm{eff}}(t) = D_1 + \sigma_0^2 [2 - \sqrt{2}]\sqrt{2\tau_D}\, t^{1/2} + O\left((t/\tau_D)^{-1/2}\right). \tag{3.6}$$

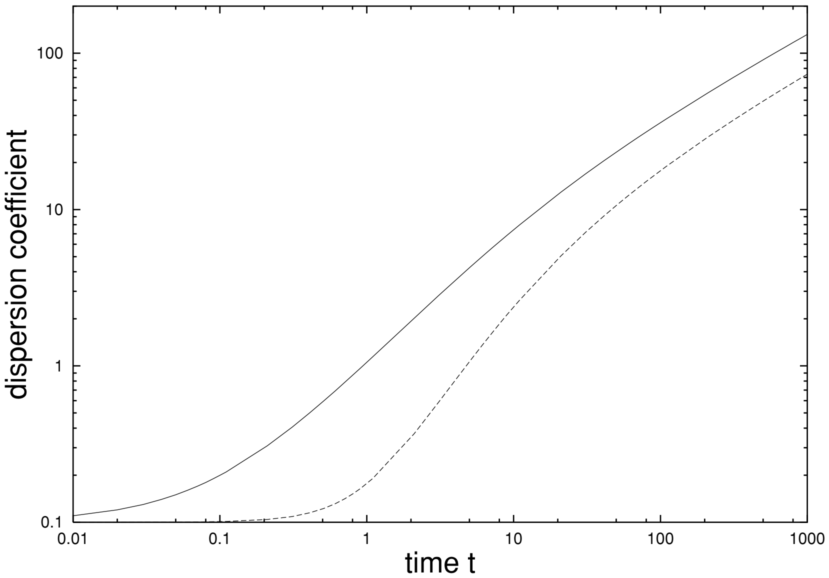

Figure 1. Ensemble (——) and effective (– – –) dispersion coefficients of an exactly stratified medium for a point source; $\tau_D = 10$, $D_1 = 0.1$, $\sigma_0^2 = 1$.

They show the well-known anomalous diffusion $t^{1/2}$ behaviour with different pre-factors for the two types of dispersion coefficients, a result which has already been described in [2, 5, 8]. These authors derive this result by considering the asymptotic behaviour of a particle exhibiting a random walk and experiencing a different velocity after each time step, i.e. in each layer. The factor of $(2-\sqrt{2})$ between $D_{11}^{\mathrm{ens}}(t)$ and $D_{11}^{\mathrm{eff}}(t)$ is due to large fluctuations in the centre-of-mass coordinates of the source distribution in different realizations.

For an extended initial distribution (in particular for the line source given by equation (2.17)) the expression for the ensemble coefficient $D_{11}^{\mathrm{ens}}(t)$ as given in (3.1) remains unchanged. The effective dispersion coefficient, however, now reads

$$D_{11}^{\mathrm{eff}}(t)|_{\mathrm{line}} = D^{\mathrm{ens}}(t) - \sigma_0^2\sqrt{\tau_D}\sqrt{\tau_D+\tau_L}\left\{\sqrt{1+\frac{4t}{\tau_D+\tau_L}} - \sqrt{1+\frac{2t}{\tau_D+\tau_L}}\right\}. \qquad (3.7)$$

Another timescale that becomes relevant, $\tau_L = L^2/D_2$, is the typical time for diffusive transport over the width L of the initial concentration distribution. For times much smaller than τ_L the effect of a finite initial extent of the injection region is equivalent to averaging over various realizations. The dispersion coefficient tends to the ensemble value given by equation (3.1). For times much longer than τ_L the system behaviour no longer depends on the initial width of the injection region. The effective dispersion coefficient, therefore, tends to the value derived for a point source (equation (3.2)). The resulting temporal behaviour is plotted in figure 2. For an infinite L the value of $D_{11}^{\mathrm{eff}}(t)|_{\mathrm{line}}$ tends to $D_{11}^{\mathrm{ens}}(t)$ for all times.

4. The general case

Unlike the exactly stratified model, the long-time behaviour for the case $v_0 \neq 0$ shows normal diffusion for asymptotically large times, $t \to \infty$. Evaluating equations (2.15) and (2.16) now

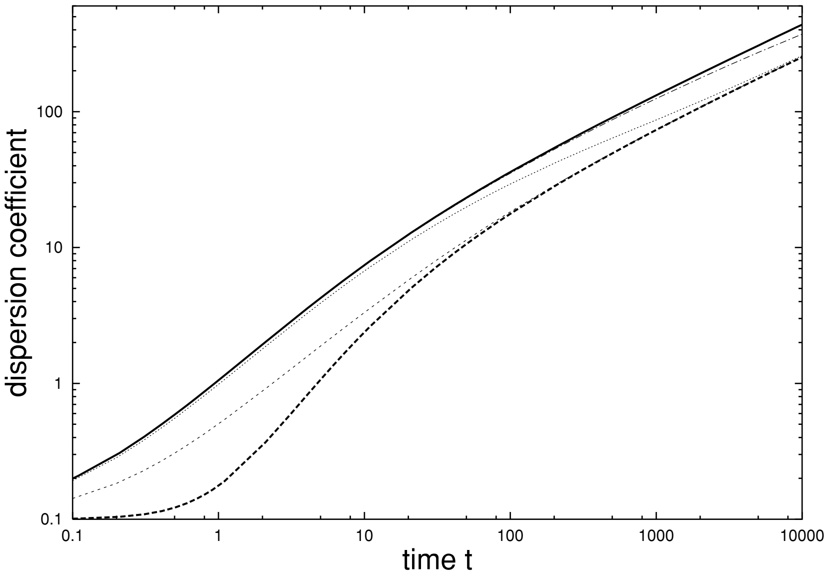

Figure 2. Ensemble (——) and effective dispersion coefficients of an exactly stratified medium for a point source (– – –) and line sources of various length ($L = 1$ (- - - -), $L = 10$ (······), $L = 100$(— · —)); $\tau_D = 10$, $D_1 = D_2 = 0.1$, $\sigma_0^2 = 1$.

leads to the more complicated expressions

$$D_{11}^{\text{ens}}(t) = D_1 + \sqrt{\frac{\pi}{2}}\sigma_0^2\tau_v \left[\text{erf}\left(\sqrt{\frac{t^2/2\tau_v^2}{1+2t/\tau_D}}\right) + \exp\left(\frac{\tau_D^2}{2\tau_v^2}\right) \times \left\{\text{erf}\left(\sqrt{\frac{t^2/2\tau_v^2}{1+2t/\tau_D}}(\tau_D/t+1)\right) - \text{erf}\left(\sqrt{\frac{\tau_D^2}{2\tau_v^2}}\right)\right\}\right] \tag{4.1}$$

$$D_{11}^{\text{eff}}(t) = D^{\text{ens}}(t) - \sqrt{\frac{\pi}{2}}\sigma_0^2\tau_v \left[\text{erf}\left(\sqrt{\frac{t^2/2\tau_v^2}{1+2t/\tau_D}}\right) + \exp\left(\frac{\tau_D^2}{2\tau_v^2}(1+4t/\tau_D)\right) \times \left\{\text{erf}\left(\sqrt{\frac{t^2/2\tau_v^2}{1+2t/\tau_D}}(\tau_D/t+3)\right) - \text{erf}\left(\sqrt{\frac{\tau_D^2}{2\tau_v^2}(1+4t/\tau_D)}\right)\right\}\right] \tag{4.2}$$

where erf (x) is the Gaussian error function as defined in [12]. The timescale $\tau_v = l/v_0$ is set by the time necessary to transport a tracer particle advectively over one correlation length of the flow field.

For $\tau_D \ll \tau_v$ three regimes can be distinguished which are shown in figure 3. For small times the advection has only a negligible influence on the dispersion coefficient. We can therefore identify the two super-diffusive regimes from the exactly stratified case, i.e. the Taylor regime given by equations (3.3) and (3.4) for $t \ll \tau_D \ll \tau_v$ (neglecting terms of the order $O(\tau_D/\tau_v)$ and $O((t/\tau_v)^2)$, respectively) and the super-diffusive $t^{1/2}$-behaviour given by equations (3.5) and (3.6) for $\tau_D \ll t \ll \tau_v$ (neglecting terms of the order $O(\tau_D/t)$ and $O((t\tau_D/\tau_v^2)^2)$, respectively). In the asymptotic limit $t \to \infty$ the system exhibits normal diffusion. The values of the dispersion coefficients are

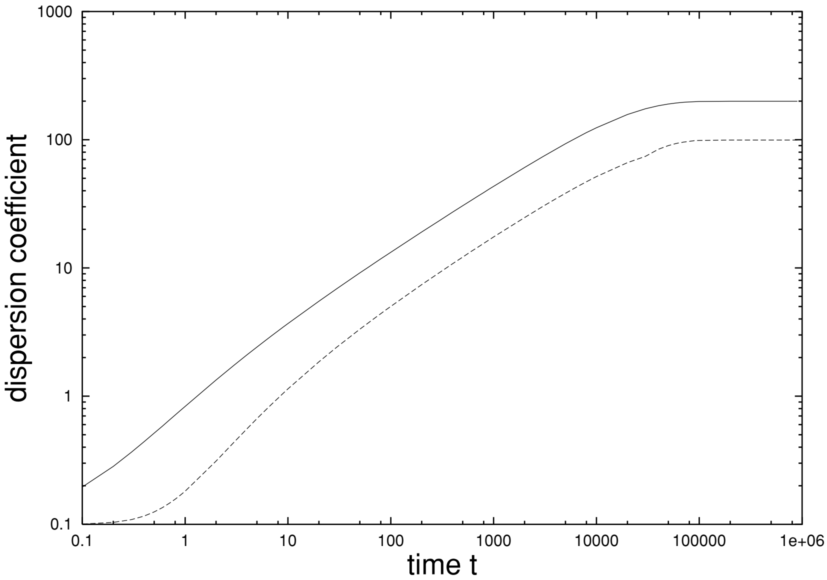

Figure 3. Ensemble (——) and effective (– – –) dispersion coefficients for a stratified medium with a flow component transverse to the strata; $\tau_D = 1$, $\tau_v = 80$, $D_1 = 0.1$, $\sigma_0^2 = 1$.

$$D_{11}^{\text{ens}}(t) = D_1 + \sqrt{2\pi}\sigma_0^2\tau_v + O(\tau_D/\tau_v) + O(\tau_v/\sqrt{t\tau_D}) \tag{4.3}$$

$$D_{11}^{\text{eff}}(t) = D_1 + \frac{1}{2}\sqrt{2\pi}\sigma_0^2\tau_v + O(\tau_D/\tau_v) + O(\tau_v/\sqrt{t\tau_D}) \tag{4.4}$$

which show that, somewhat unexpectedly, the two quantities do not become equal even in the normal diffusive case. This model seems to be a counter-example to Metzger's conjecture [13] that for normally diffusive models the effective and the ensemble dispersion coefficients should always become equal in the asymptotic limit. One remarkable feature of the crossover between the super-diffusive and the normal diffusive regime is that it does not occur at τ_v, but at a time scale $\tau_v^2/\tau_D = D_2/v_0^2$ which is independent of the correlation length of the system.

For the case $\tau_v \ll \tau_D$ the behaviour is much simpler. Advection dominates over diffusion, i.e. the diffusive timescale τ_D cannot be resolved. Therefore there are only two regimes: for $t \ll \tau_v$ we again find the Taylor regime with a linear growth of the ensemble dispersion coefficient; at τ_v there is a crossover to a normal diffusive regime given by

$$D_{11}^{\text{ens}}(t) = D_1 + \sqrt{\frac{\pi}{2}}\sigma_0^2\tau_v + O\left(\frac{\tau_D}{t}\right) + O\left(\sqrt{\frac{\tau_v}{\tau_D}}\right). \tag{4.5}$$

Because the diffusive effect on the mixing is suppressed by advection, the effective dispersion coefficient is equal to D_1 in leading order:

$$D_{11}^{\text{ens}}(t) = D_1 + O\left(\frac{\tau_D}{t}\right) + O\left(\sqrt{\frac{\tau_v}{\tau_D}}\right). \tag{4.6}$$

5. Self-averaging

The question that remains is whether the effective ensemble-averaged parameters represent typical behaviour in a single realization of the statistical ensemble. To investigate this problem further, we calculate the root-mean-square sample-to-sample fluctuations of the centre-of-mass velocity and the dispersion coefficient which are given by

$$(\delta u(t))^2 \equiv \overline{u(t)^2} - \overline{u(t)}^2 \tag{5.1}$$

$$(\delta D(t))^2 \equiv \overline{D(t)^2} - \overline{D(t)}^2 \tag{5.2}$$

where $u(t)$ and $D(t)$ stand for the centre-of-mass velocity and the dispersion coefficient of the single realizations as defined in (2.9) and (2.10).

Again, these quantities can be evaluated explicitly in the given model. As expected, we find that the effective velocity becomes self-averaging in the asymptotic long-time limit, $\delta u(t) \to 0$ for $t \to \infty$. In the opposite short-time case, $t \to 0$, the fluctuations are determined by the fluctuation strength of the disorder σ_0^2. As an example, consider the simple case of an exactly stratified medium, $v_0 = 0$, with point-like injection (i.e. $L = 0$). The explicit temporal behaviour of the root-mean-square fluctuations is found to be

$$\frac{(\delta u(t))^2}{\overline{u(t)}^2} = \frac{(\delta u(t))^2}{u_0{}^2} = \frac{\sigma_0^2}{u_0^2}\sqrt{\frac{1}{1+4t/\tau_D}}. \tag{5.3}$$

Surprisingly, the situation is different in the case of the dispersion coefficient. It does not become self-averaging even in the asymptotic long-time limit. Here, in the limit $t \to \infty$ the normalized fluctuations $\delta D/\overline{D}^2$ tend to a constant finite value for the exactly stratified case $v_0 = 0$ as well as for the case $v_0 \neq 0$ where one has normal diffusion in this limit.

The explicit results are somewhat cumbersome and not very illustrative. The derivation and the explicit formulae are, therefore, deferred to appendix B. Figure 4 illustrates the qualitative behaviour for the case of an exactly stratified medium (i.e. $v_0 = 0$) and line injections of different lengths L. As already described in the previous two sections, the short-time regime is characterized by an effective dispersion coefficient which is mainly determined by local diffusion. This quantity does not vary between different realizations of the flow field; the fluctuations therefore obviously vanish for $t \to 0$. For $t \to \infty$ the sample-to-sample fluctuations tend to a constant value, irrespective of the initial length of the injection source. The timescale on which this asymptotic value is reached is given by $\tau_D + \tau_L$, i.e. it is large for larger L. Consequently, an infinitely long source will show no fluctuations at all as all the fluctuations of the flow field are already experienced right from the start.

The non-vanishing sample-to-sample fluctuations demonstrate the fact that the effective dispersion coefficient as defined by (2.12) is not a self-averaging quantity; it therefore does not represent the typical behaviour expected in a single representation of the flow field in the two-dimensional stratified model.

Appendix A

In this appendix, we sketch the basic steps used to derive the results for the temporal behaviour of the transport coefficients discussed in the text. The starting point is the solution of the Langevin equation given by (2.5) and (2.6) which is inserted into equation (2.8) with $n = 1$ and $n = 2$. To perform the necessary white-noise averages, it is useful to replace the flow field by its Fourier representation, $u'(x_2(t)) = \int_k \tilde{u}'(k)\ \mathrm{e}^{-\mathrm{i}k\,x_2(t)}$. After inserting (2.5) and (2.6), the resulting white-noise averages can be calculated using the general formula $\langle \exp\{-\mathrm{i}\sum_j \int \mathrm{d}t' L_j(t')\,\xi_j(t')\}\rangle = \exp\{-\sum_{j,l}\int \mathrm{d}t' L_j(t')\,D_{jl}\,L_l(t')\}$ and an appropriate choice of the arbitrary auxiliary function $L_j(t)$. One ends up with

$$\langle x_1(t)|_{\mathbf{x}_0}\rangle = x_{01} + u_0 t + \int_0^t \mathrm{d}t' \int_k \tilde{u}'(k)\mathrm{e}^{\mathrm{i}k(x_{02}+v_0 t')}\mathrm{e}^{-k^2 D_2 t'} \tag{A.1}$$

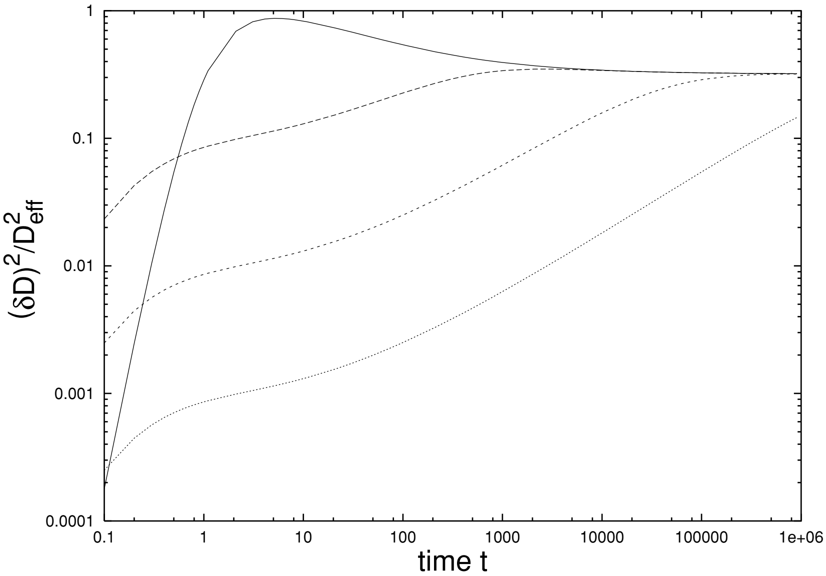

Figure 4. Normalized fluctuations of the dispersion coefficient in an exactly stratified medium for a point source (——) and line sources with $L = 10$ (– – –), $L = 100$ (- - - -), $L = 1000$ (· · · · · ·); $\tau_D = 10$, $D_1 = D_2 = 0.1$, $\sigma_0^2 = 1$.

$$\begin{aligned}\langle x_1^2(t)|_{\mathbf{x}_0}\rangle &= 2D_1 t + (x_{01} + u_0 t)^2 + 2\,(x_{01} + u_0 t)\int_0^t \mathrm{d}t' \int_k \tilde{u}'(k)\mathrm{e}^{\mathrm{i}k(x_{02}+v_0 t')}\mathrm{e}^{-k^2 D_2 t'}\\ &\quad + 2\int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \int_k \int_{k'} \tilde{u}'(k)\tilde{u}'(k')\mathrm{e}^{-\mathrm{i}k(x_{02}+v_0 t')-\mathrm{i}k'(x_{02}+v_0 t'')}\\ &\quad \times \exp\left(-k^2 D_2 t' - k'^2 D_2 t'' - 2kk' D_2 \min(t', t'')\right)\end{aligned} \tag{A.2}$$

and

$$\langle x_2(t)|_{\mathbf{x}_0}\rangle = x_{02} + v_0 t \tag{A.3}$$

$$\langle x_2^2(t)|_{\mathbf{x}_0}\rangle = 2D_2 t + (x_{02} + v_0 t)^2\,. \tag{A.4}$$

These expressions still depend on the initial positions $\mathbf{x}_0$ of the particles. According to (2.8) one finally performs the remaining average over the initial distribution $\rho(\mathbf{x}_0)$. Inserted into the definitions for the effective and ensemble dispersion coefficients (2.12) and (2.13), and performing the ensemble average over the disordered flow field, this eventually leads to (2.14), (2.15) and (2.16). Using the explicit form for the initial distribution of equation (2.17) and the expression for the disorder averages (2.3) and (2.4) then yields the final explicit results given in the text.

Appendix B

In appendix B, we sketch the basic steps used to derive the results for the sample-to-sample fluctuations of the transport coefficients given in the text.

For an exactly stratified medium (i.e. $v_0 = 0$) and a point-like injection (i.e. $L = 0$), using the steps described in appendix A, one derives the following results for the square of the dispersion coefficient $D(t)$ in a given realization of the medium as defined by (2.10)

$$D(t)^2 = D_1^2 + 2D_1 \int_{k_1}\int_{k_2} \tilde{u}(k_1)\tilde{u}(k_2) \int_0^t \mathrm{d}t_1 \mathrm{e}^{-k_1^2 D_2 t - k_2^2 D_2 t_1} \left(\mathrm{e}^{-2k_1k_2D_2t_1} - 1\right)$$
$$+ \int_{k_1}\int_{k_2}\int_{k_3}\int_{k_4} \tilde{u}(k_1)\tilde{u}(k_2)\tilde{u}(k_3)\tilde{u}(k_4) \times \mathrm{e}^{-k_1^2 D_2 t - k_2^2 D_2 t_1}$$
$$\times \left(\mathrm{e}^{-2k_1k_2D_2t_1} - 1\right) \mathrm{e}^{-k_3^2 D_2 t - k_4^2 D_2 t_2} \left(\mathrm{e}^{-2k_3k_4D_2t_2} - 1\right). \tag{B.1}$$

Assuming a Gaussian ensemble for the flow field, the four-point function $\overline{\tilde{u}(k_1)\tilde{u}(k_2)\tilde{u}(k_3)\tilde{u}(k_4)}$ factorizes as usual into products of two-point correlation functions. Using this decomposition and a rescaling of the time integrations of the form $\tau_1 := t_i/t$ we get for the root-mean-square sample-to-sample fluctuations of the dispersion coefficient defined by (5.2)

$$(\delta D(t))^2 = \frac{1}{2}\sigma_0^4 \tau_D t \int_0^1 \mathrm{d}\tau_1 \int_0^1 \mathrm{d}\tau_2 \left\{ A\left(\frac{\tau_D}{2t}, \tau_1, \tau_2\right)^{-1/2} + B\left(\frac{\tau_D}{2t}, \tau_1, \tau_2\right)^{-1/2} \right.$$
$$- 2\left[A\left(\frac{\tau_D}{2t}, \tau_1, \tau_2\right) - \tau_2^2\right]^{-1/2} - 2\left[B\left(\frac{\tau_D}{2t}, \tau_1, \tau_2\right) - \tau_2^2\right]^{-1/2}$$
$$\left. + \left[A\left(\frac{\tau_D}{2t}, \tau_1, \tau_2\right) - (\tau_1+\tau_2)^2\right]^{-1/2} + \left[B\left(\frac{\tau_D}{2t}, \tau_1, \tau_2\right) - (\tau_1+\tau_2)^2\right]^{-1/2} \right\} \tag{B.2}$$

with

$$A\left(\frac{\tau_D}{2t}, \tau_1, \tau_2\right) := \left(\frac{\tau_D}{2t} + 2\right)\left(\frac{\tau_D}{2t} + \tau_1 + \tau_2\right)$$
$$B\left(\frac{\tau_D}{2t}, \tau_1, \tau_2\right) := \left(\frac{\tau_D}{2t} + 1 + \tau_1\right)\left(\frac{\tau_D}{2t} + 1 + \tau_2\right). \tag{B.3}$$

The calculation is easily generalized to the case of an extended injection (i.e. $L \neq 0$). There, the functions $A(\frac{\tau_D}{2t}, \tau_1, \tau_2)$ and $B(\frac{\tau_D}{2t}, \tau_1, \tau_2)$ are replaced by

$$A\left(\frac{\tau_D+\tau_L}{2t}, \tau_1, \tau_2\right) := \left(\frac{\tau_D+\tau_L}{2t} + 2\right)\left(\frac{\tau_D+\tau_L}{2t} + \tau_1 + \tau_2\right)$$
$$B\left(\frac{\tau_D+\tau_L}{2t}, \tau_1, \tau_2\right) := \left(\frac{\tau_D+\tau_L}{2t} + 1 + \tau_1\right)\left(\frac{\tau_D+\tau_L}{2t} + 1 + \tau_2\right). \tag{B.4}$$

For a medium with a drift component perpendicular to the strata (i.e. $v_0 \neq 0$) and a point-like injection (i.e. $L = 0$) , we get

$$(\delta D(t))^2 = \int_{k_1}\int_{k_1} C(k_1)C(k_2)\left[\mathrm{e}^{-2k_1^2D_2t} F(k_1,k_2)F^*(k_1,k_2)\right.$$
$$\left. + \mathrm{e}^{-\mathrm{i}v_0(k_1-k_2)t}\mathrm{e}^{-\left(k_1^2+k_2^2\right)D_2t} F(k_1,k_2)F^*(k_2,k_1)\right] \tag{B.5}$$

where $C(k)$ is the partially integrated correlation function

$$C(k) := \int_{k_1'}\int_{k_2'}\int_{k_1} \overline{\tilde{u}(\mathbf{k})\tilde{u}(\mathbf{k}')} = \sqrt{2\pi}\sigma_0^2 l \exp\left(-\frac{k^2l^2}{2}\right) \tag{B.6}$$

and

$$F(k_1,k_2) = \frac{1}{\mathrm{i}v_0k_2 + k_2^2D_2}\left(\mathrm{e}^{-\left(\mathrm{i}v_0k_2+k_2^2D_2\right)t} - 1\right)$$
$$- \frac{1}{\mathrm{i}v_0k_2 + k_2^2D_2 + 2k_1k_2D_2}\left(\mathrm{e}^{-\left(\mathrm{i}v_0k_2+k_2^2D_2+2k_1k_2D_2\right)t} - 1\right). \tag{B.7}$$

Rescaling the integration variable $k_i = q_i/\sqrt{t}$ and expanding the resulting expression to $O(1/\sqrt{t})$ gives

$$F(q_1/\sqrt{t}, q_2/\sqrt{t}) = \frac{1}{v_0 q_2}\left[\left(1 - \frac{q_2 D_2}{i v_0 \sqrt{t}}\right)\left(\mathrm{e}^{-\left(\mathrm{i}v_0 q_2 \sqrt{t} + q_2^2 D_2\right)} - 1\right)\right.$$
$$\left. - \left(1 - \frac{q_2 D_2}{\mathrm{i}v_0\sqrt{t}} - \frac{2q_1 D_2}{\mathrm{i}v_0\sqrt{t}}\right)\left(\mathrm{e}^{-\left(\mathrm{i}v_0 q_2\sqrt{t} + q_2^2 D_2 + 2q_1 q_2 D_2\right)} - 1\right)\right]. \tag{B.8}$$

The limit $t \to \infty$ leads to the expression

$$(\delta D(t))^2 = \frac{1}{u_2}\int_{q_1}\int_{q_2} C(0)C(0)\left(\frac{1}{q_1 q_2} + \frac{1}{q_2^2}\right)$$
$$\times\, \mathrm{e}^{\left(q_1^2 + q_2^2\right)D_2}\,\mathrm{e}^{(q_1+q_2)^2 D_2}\left(\mathrm{e}^{-q_1 q_2 D_2} - \mathrm{e}^{q_1 q_2 D_2}\right)^2 + O\left(\frac{1}{\sqrt{t}}\right). \tag{B.9}$$

The case of finite L can be treated by introducing a factor $\exp(-\frac{1}{2}\frac{L^2}{t}(q_1^2 + q_1^2))$ in the integral and replacing $F(q_1/\sqrt{t}, q_2/\sqrt{t})$ by

$$G(q_1/\sqrt{t}, q_2/\sqrt{t}) = \frac{1}{\mathrm{i}v_0 q_2 + q_2^2 D_2/\sqrt{t}}\left(\mathrm{e}^{-\left(\mathrm{i}v_0 q_2\sqrt{t} + q_2^2 D_2\right)} - 1\right)$$
$$- \frac{\mathrm{e}^{-q_1 q_2 L^2/t}}{\mathrm{i}v_0 q_2 + q_2^2 D_2/\sqrt{t} + 2q_1 q_2 D_2/\sqrt{t}}\left(\mathrm{e}^{-\left(\mathrm{i}v_0 q_2\sqrt{t} + q_2^2 D_2 + 2q_1 q_2 D_2\right)} - 1\right). \tag{B.10}$$

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